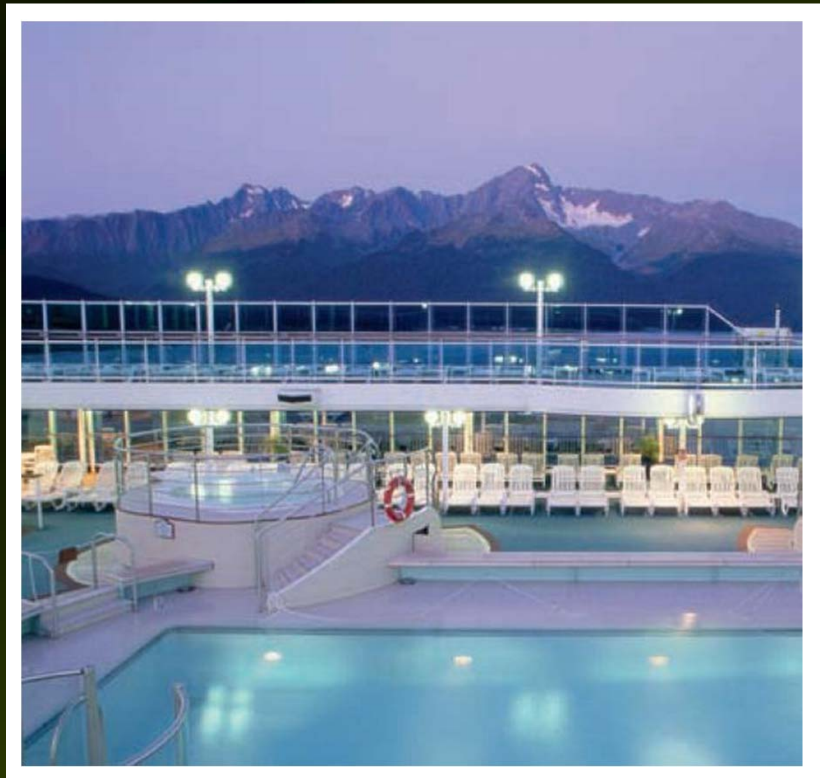
8 CALCULUS OF SEVERAL VARIABLES



8.3 Maxima and Minima of Functions of Several Variables

Relative Extrema of a Function of Two Variables

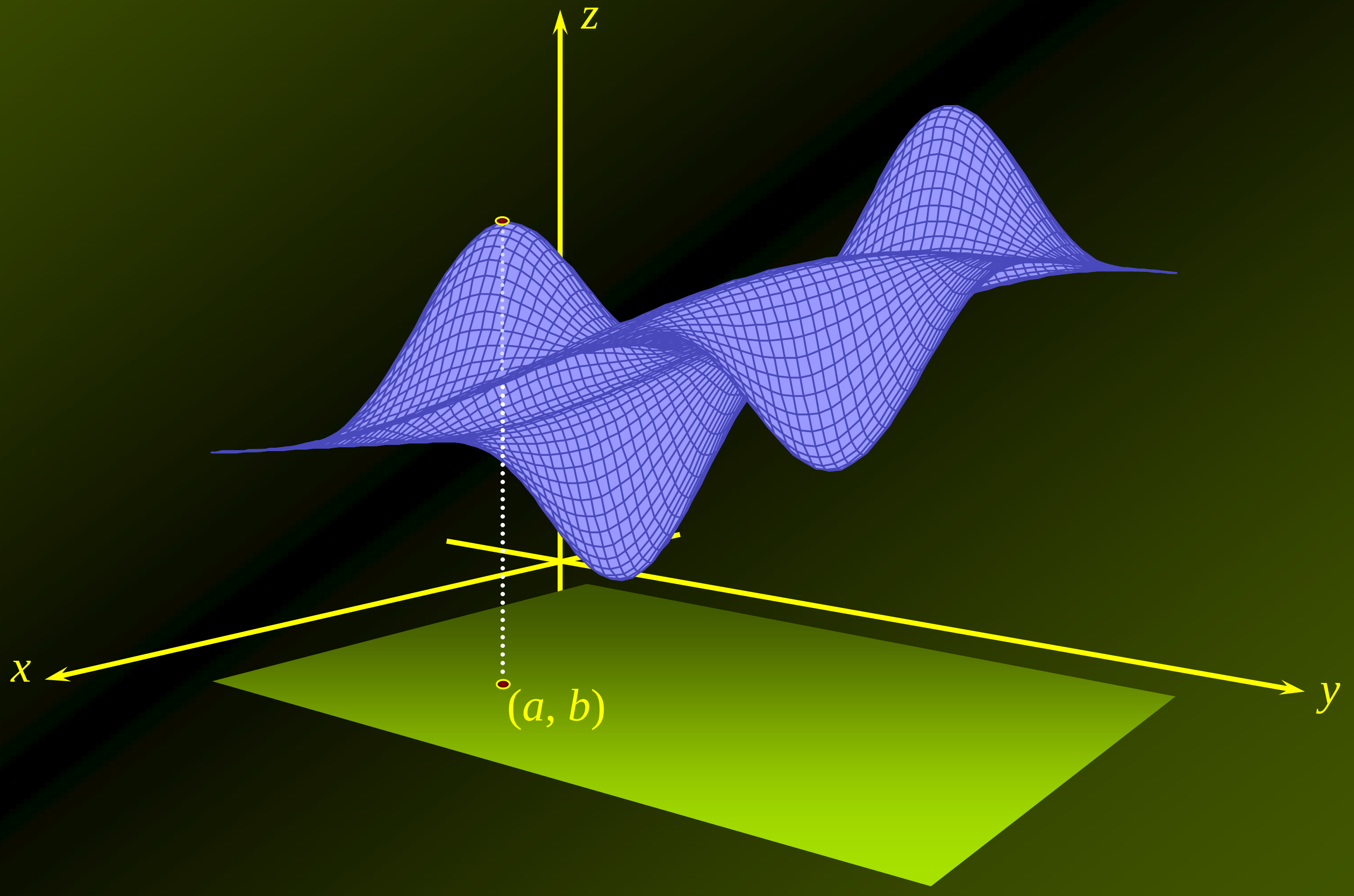
Let f be a function defined on a region R containing the point (a, b).

Then, f has a relative maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) that are sufficiently close to (a, b). The number $f(a, b)$ is called a relative maximum value.

Similarly, f has a relative minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) that are sufficiently close to (a, b). The number $f(a, b)$ is called a relative minimum value.

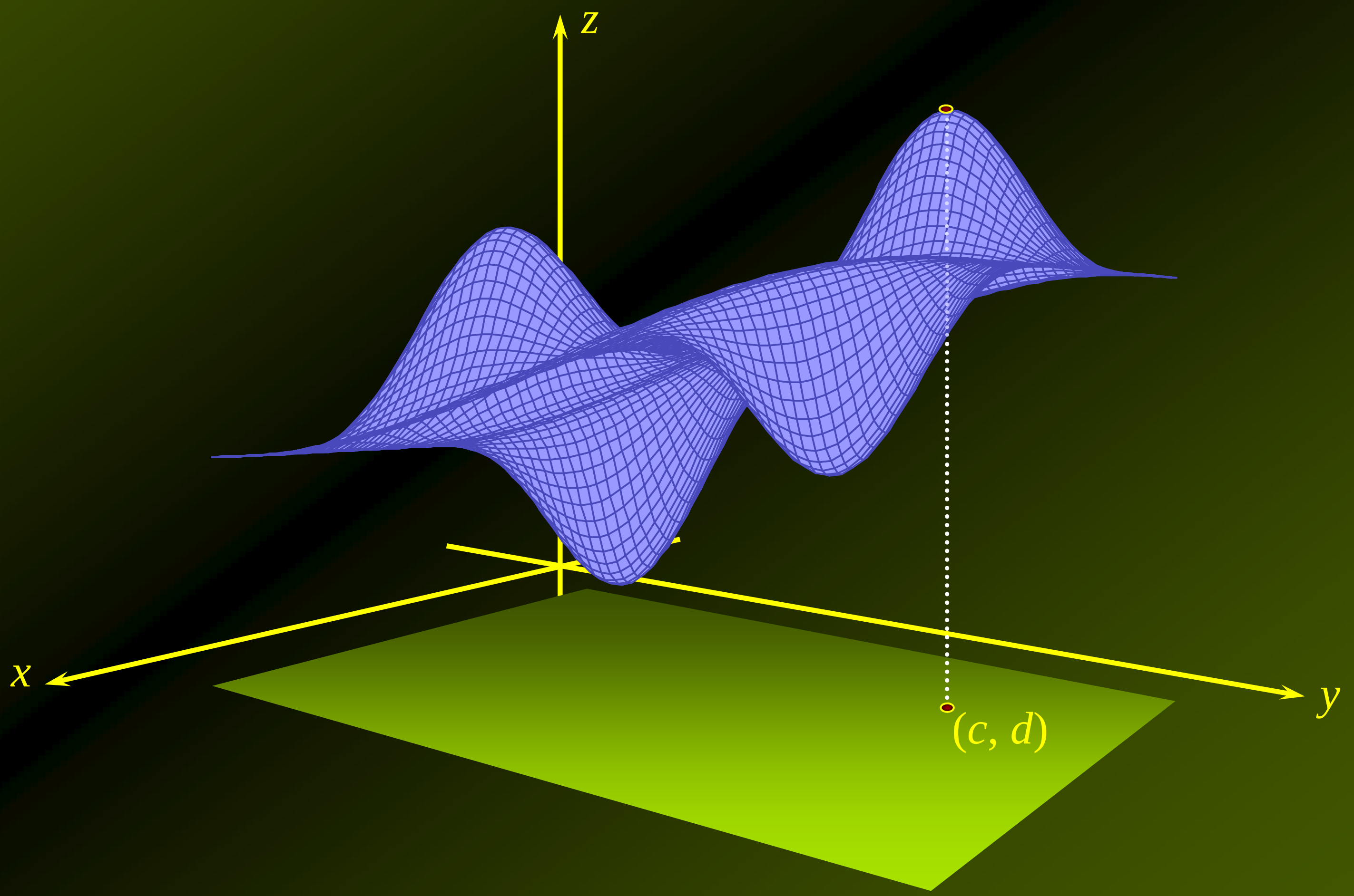
Graphic Example

There is a relative maximum at (a, b).



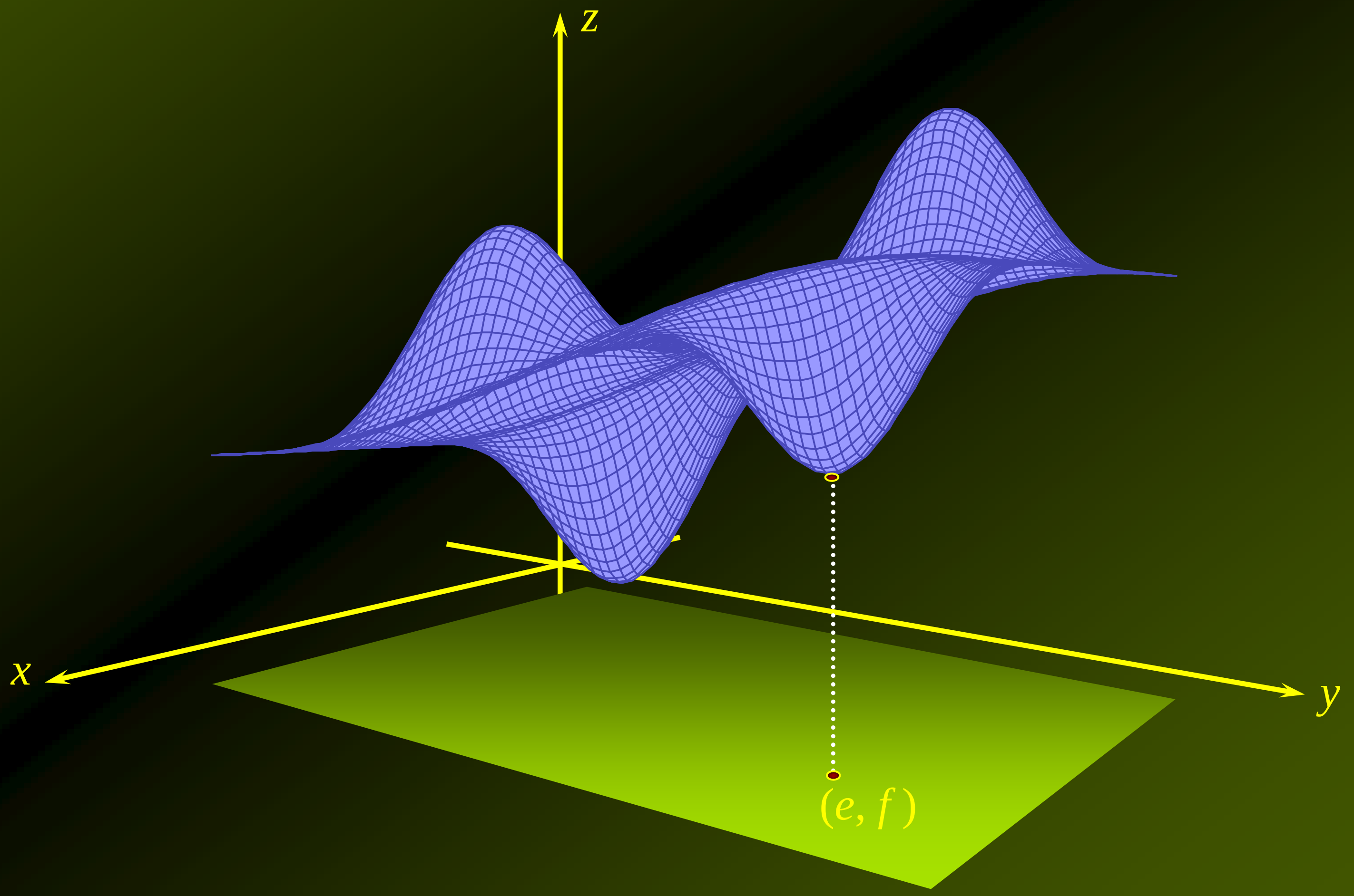
Graphic Example

There is an absolute maximum at (*c*, *d*). (It is also a relative maximum)



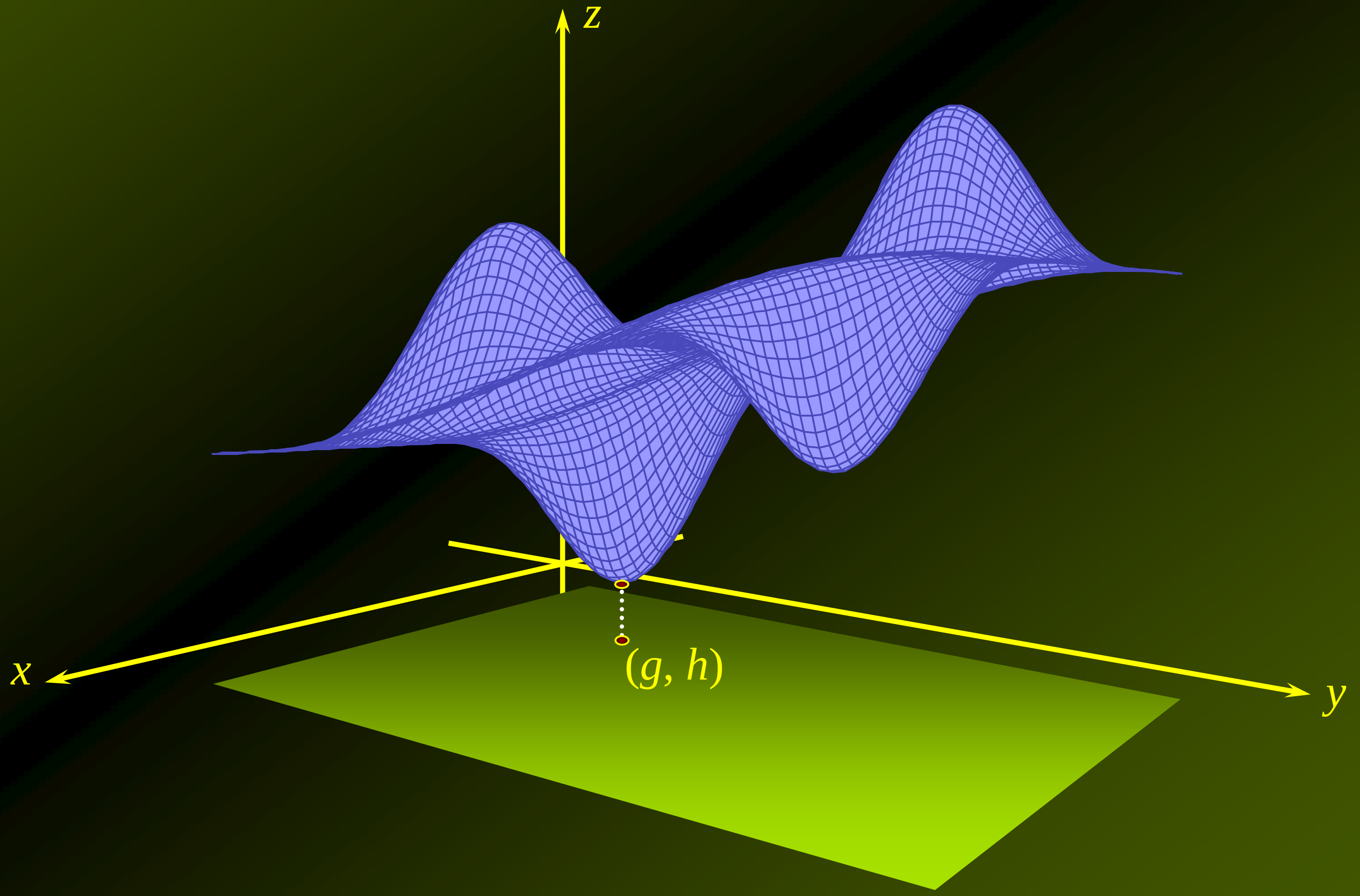
Graphic Example

There is a relative minimum at (*e*, *f*).



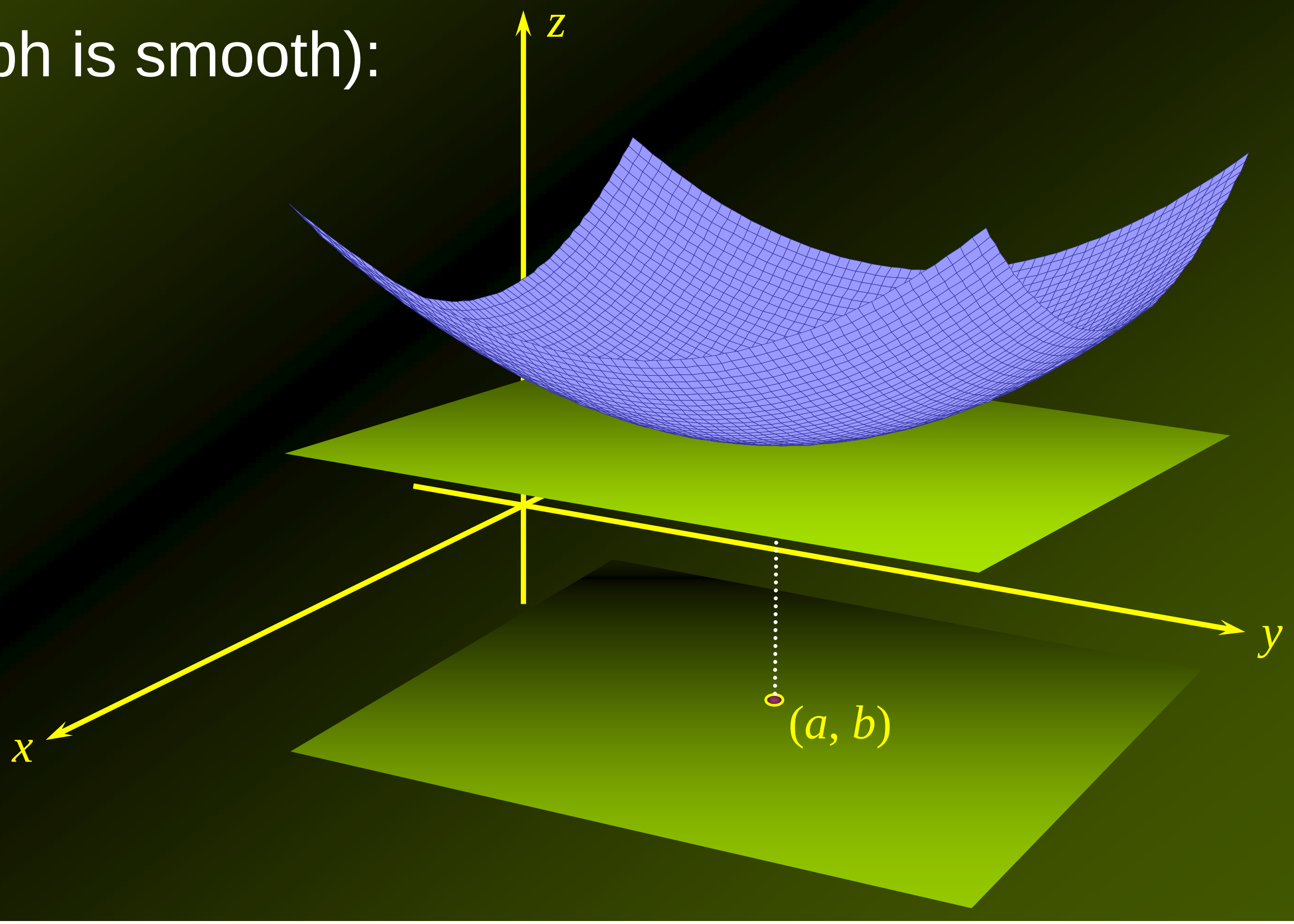
Graphic Example

There is an absolute minimum at (g, h). (It is also a relative minimum)



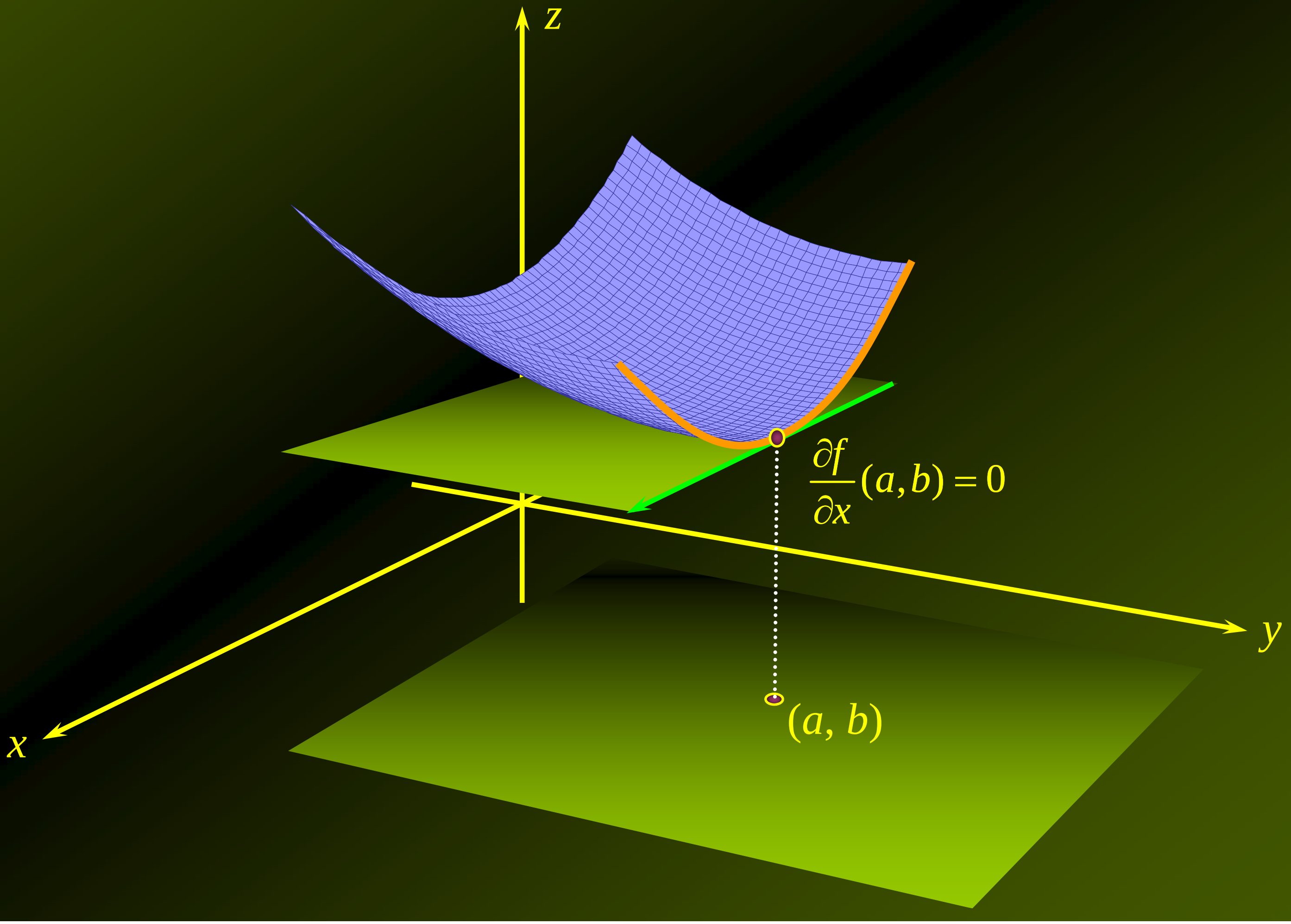
Relative Minima

At a minimum point of the graph of a function of two variables, such as point (*a*, *b*) below, the plane tangent to the graph of the function is horizontal (assuming the surface of the graph is smooth):



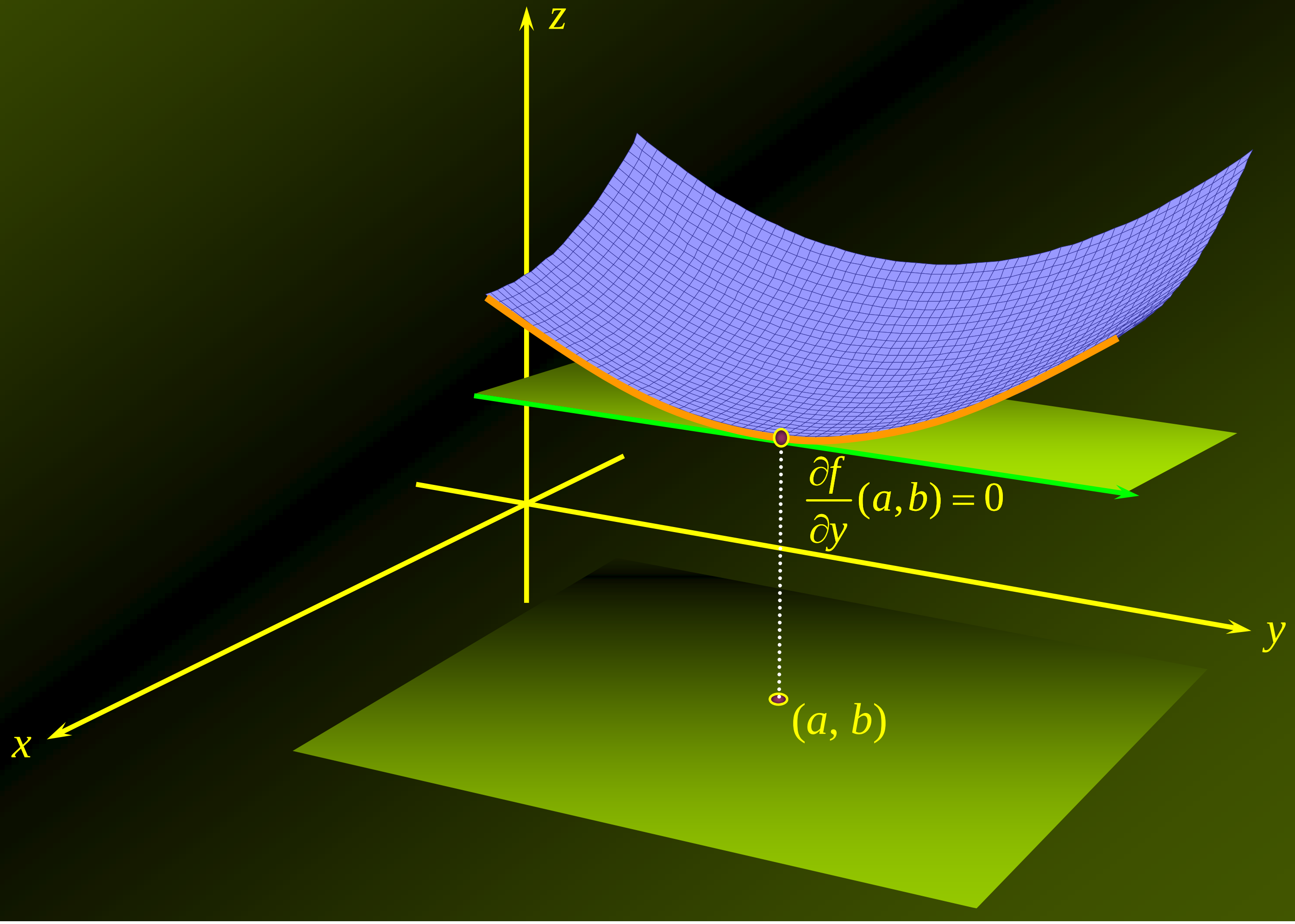
Relative Minima

Thus, at a minimum point, the graph of the function has a slope of zero along a direction parallel to the *x*-axis:



Relative Minima

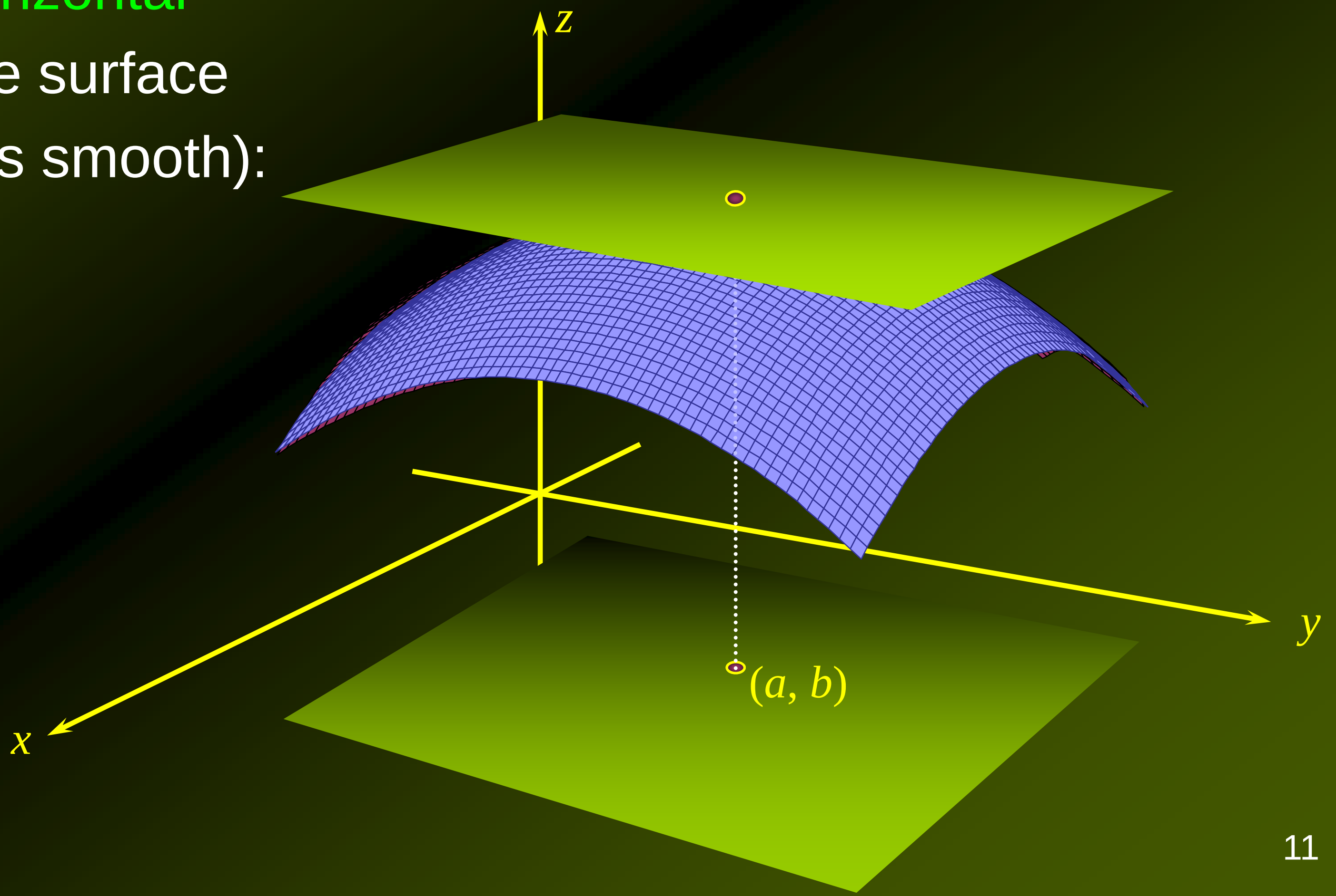
Similarly, at a minimum point, the graph of the function has a slope of zero along a direction parallel to the *y*-axis:



Relative Maxima

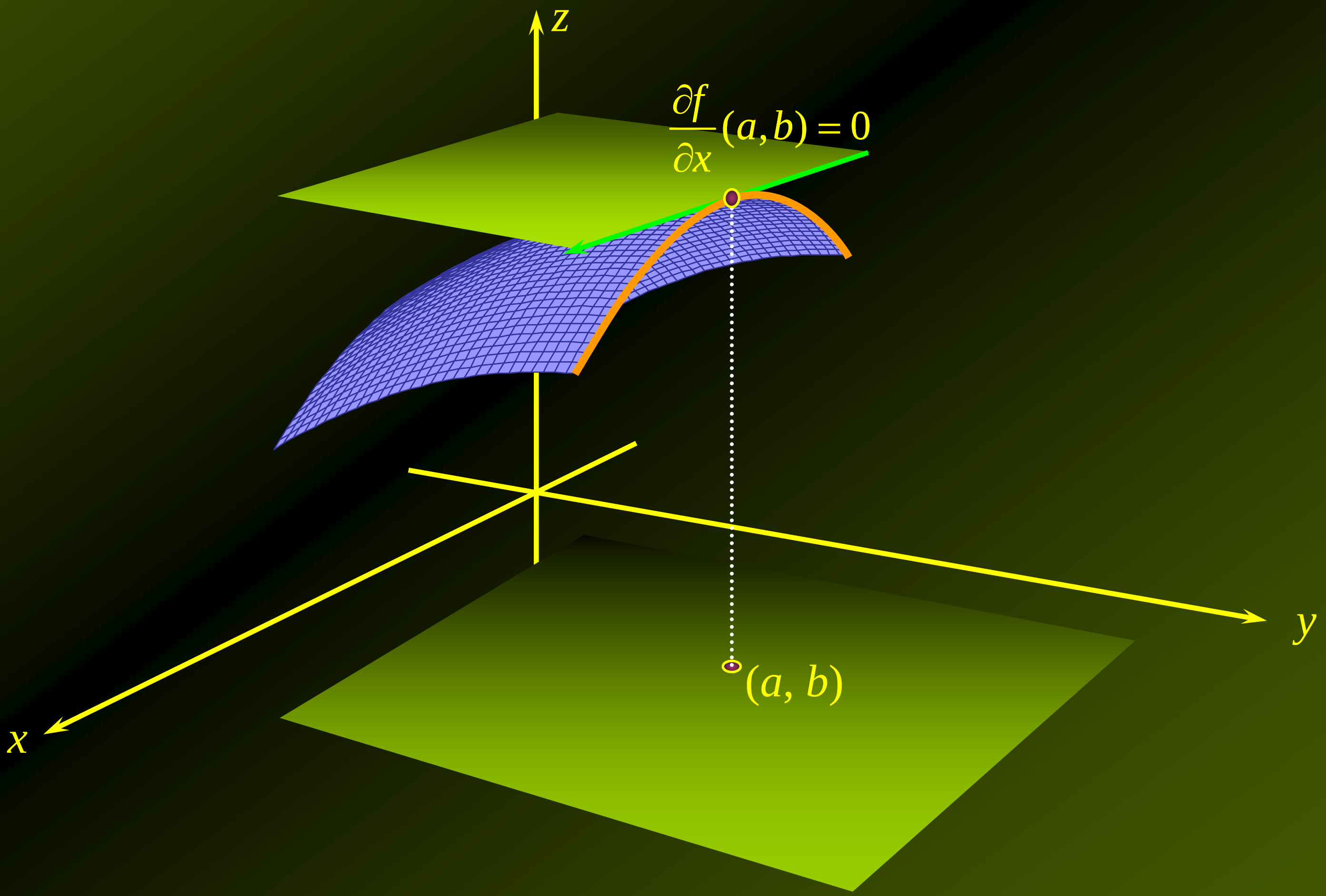
At a maximum point of the graph of a function of two variables, such as point (*a*, *b*) below, the plane tangent to the graph of the function is horizontal

(assuming the surface of the graph is smooth):



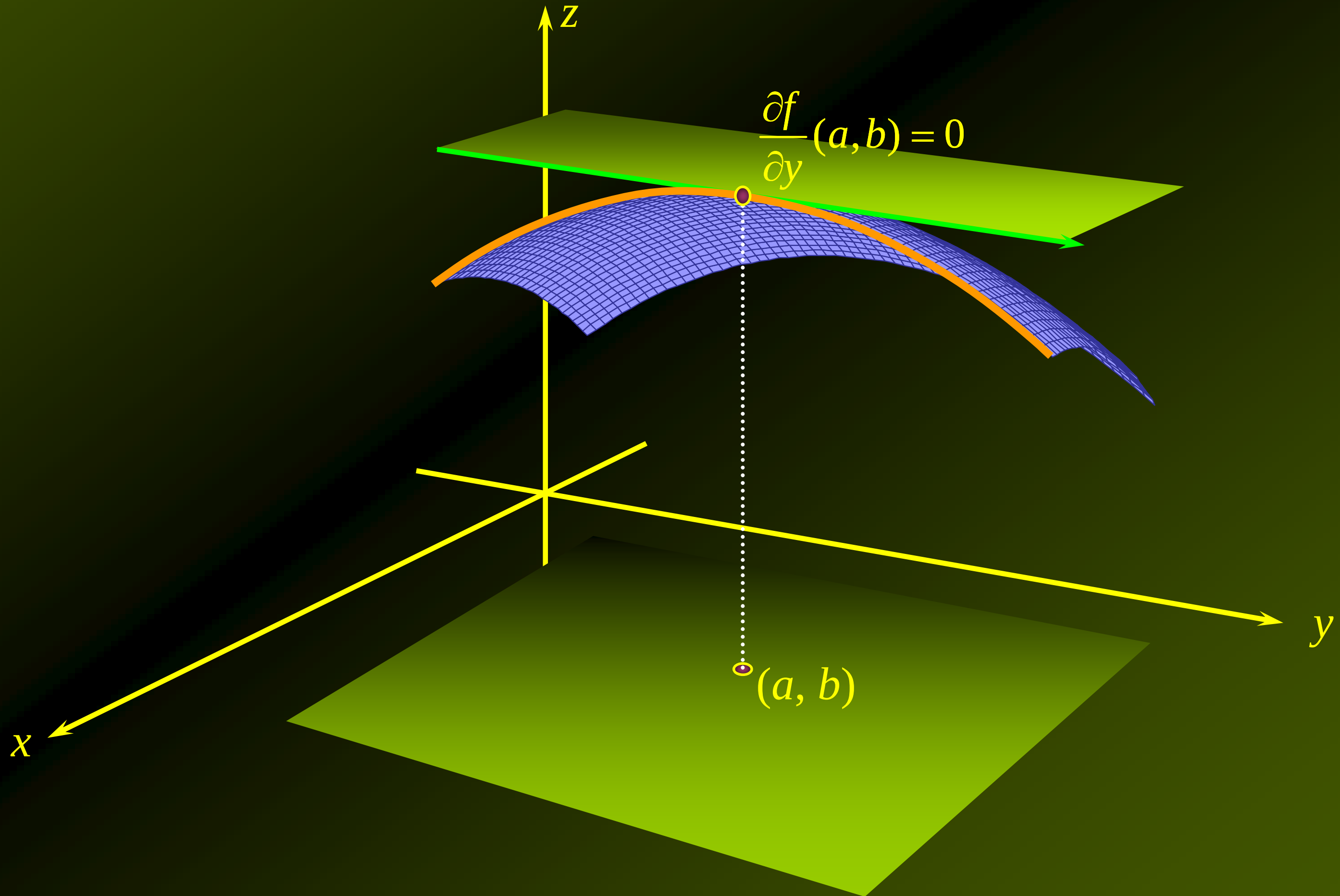
Relative Maxima

Thus, at a maximum point, the graph of the function has a slope of zero along a direction parallel to the *x*-axis:



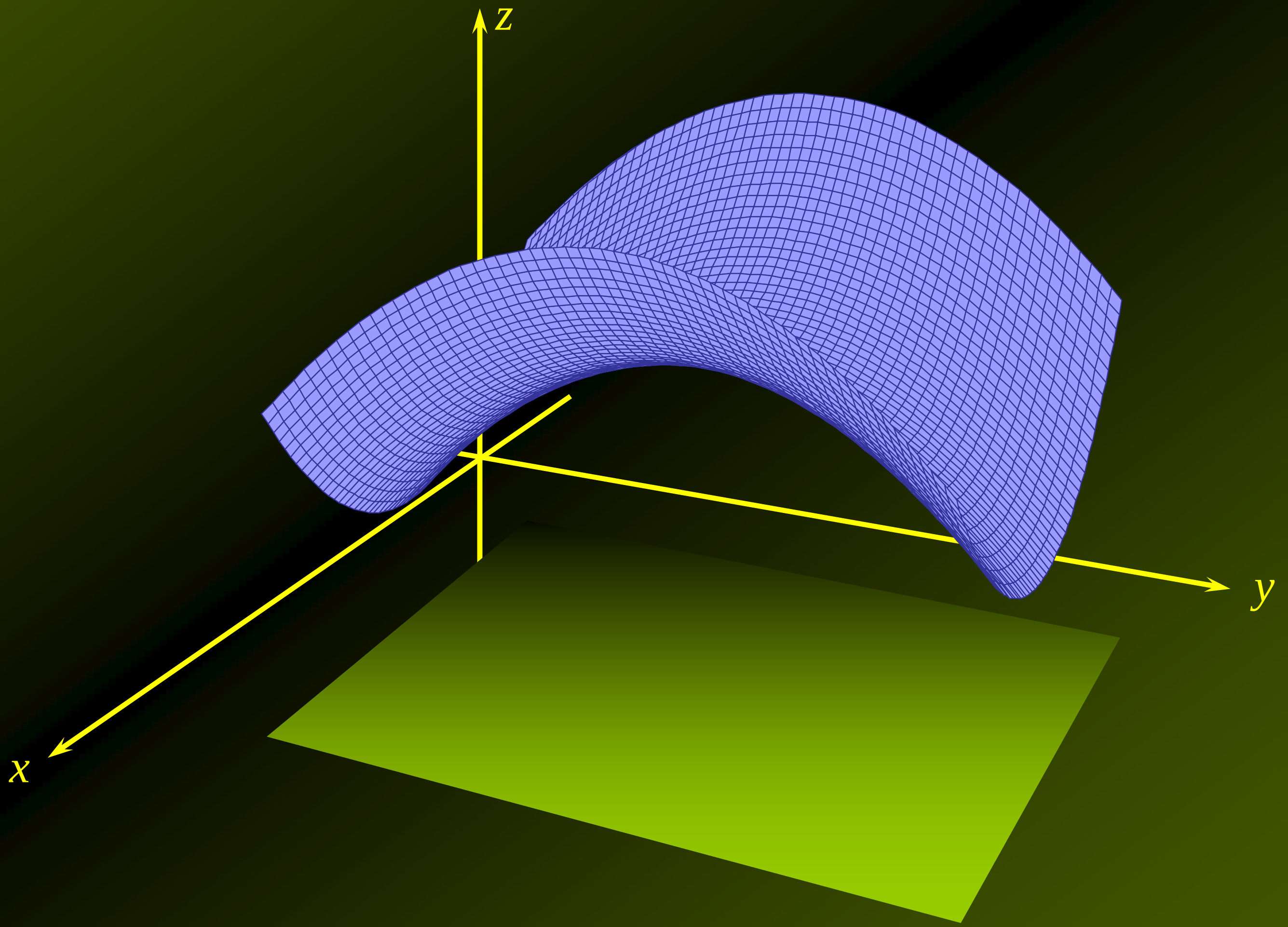
Relative Maxima

Similarly, at a maximum point, the graph of the function has a slope of zero along a direction parallel to the *y*-axis:



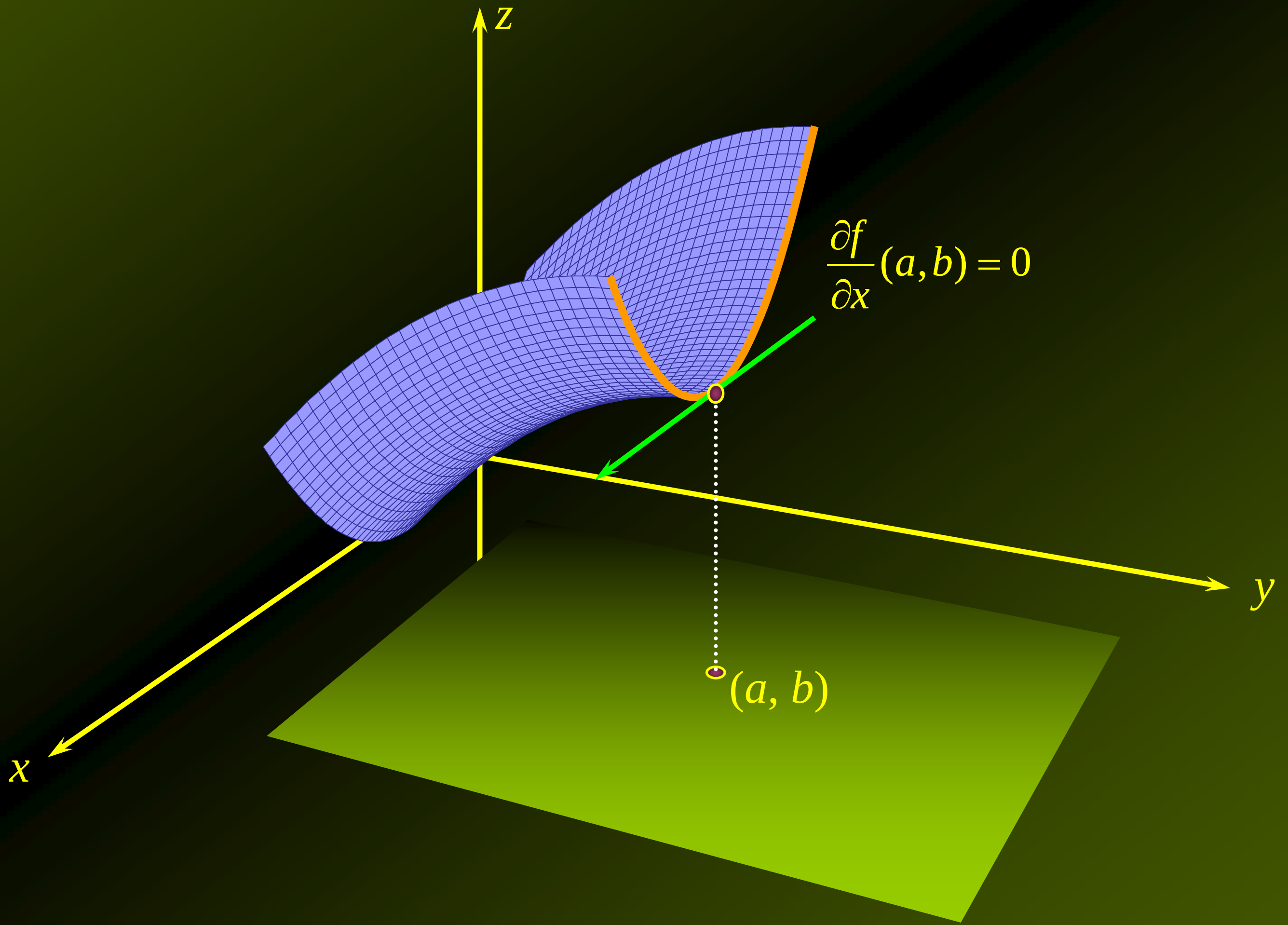
Saddle Point

In the case of a saddle point, both partials are equal to zero, but the point is neither a maximum nor a minimum.



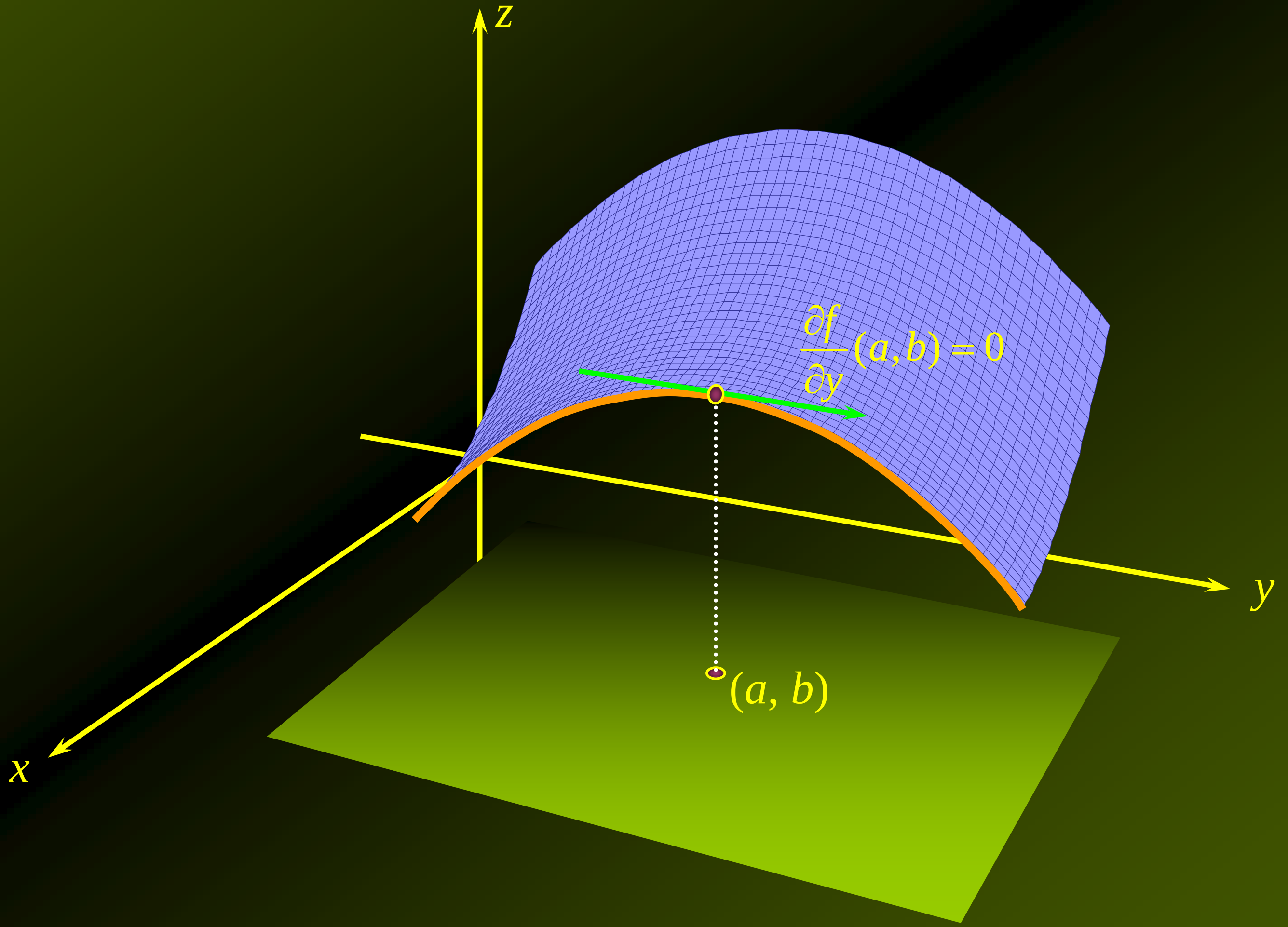
Saddle Point

In the case of a saddle point, the function is at a minimum along one vertical plane.



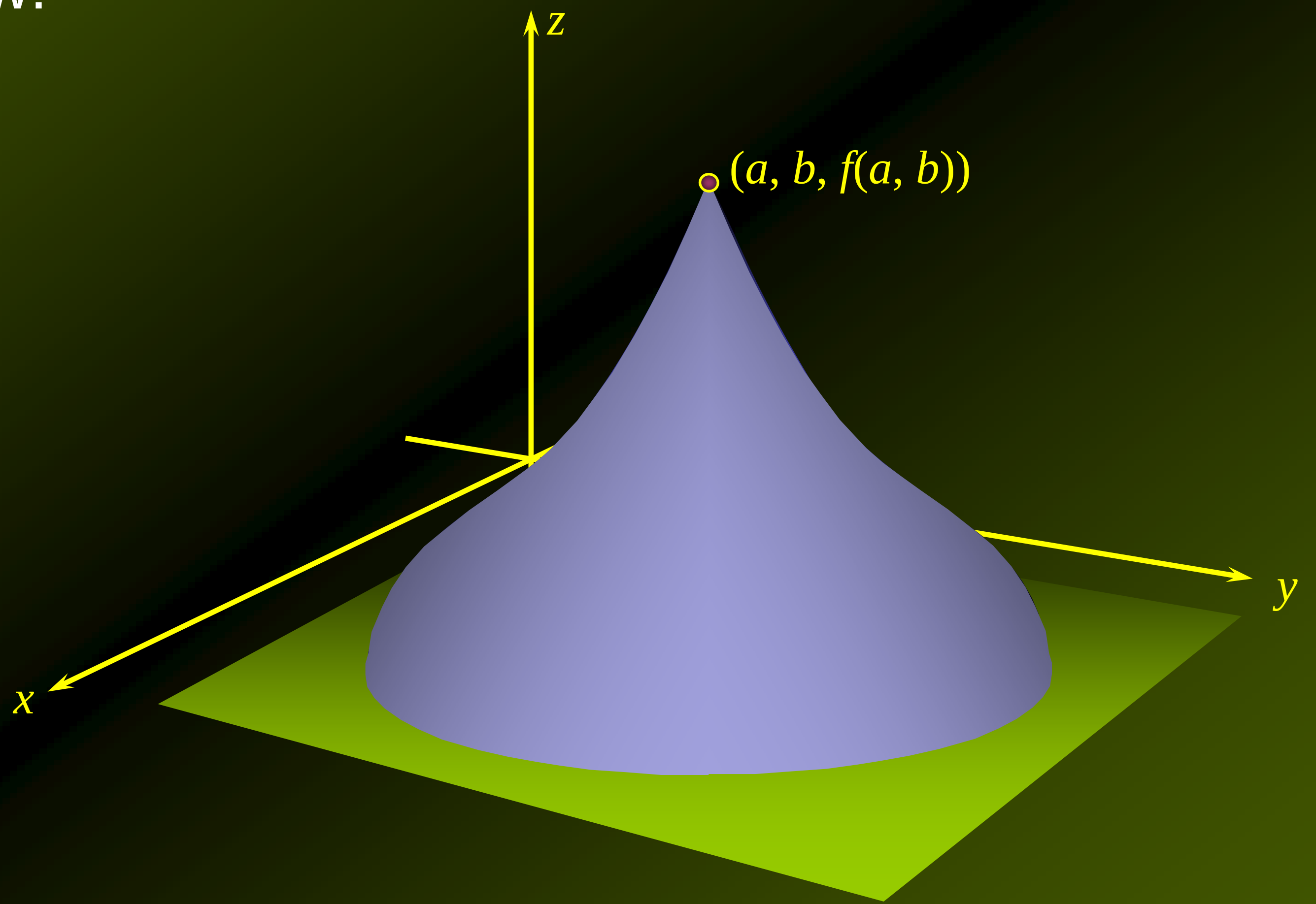
Saddle Point

In the case of a saddle point, the function is at a maximum along the perpendicular vertical plane.



Extrema When Partial Derivatives are Not Defined

A maximum (or minimum) may also occur when both partial derivatives are not defined, such as point (a, b) in the graph below:



Critical Point of a Function

A critical point of *f* is a point (*a*, *b*) in the domain of *f* such that both

$$\frac{\partial f}{\partial x}(a,b)=0 \quad \textbf{and} \quad \frac{\partial f}{\partial y}(a,b)=0$$

or at least one of the partial derivatives does not exist.

Determining Relative Extrema

1. Find the critical points of $f(x, y)$ by solving the system of simultaneous equations

$$f_x = 0 \qquad\qquad f_y = 0$$

2. The second derivative test: Let

$$D(x, y) = f_{xx} f_{yy} - f^2_{xy}$$

3. Then,
   a. $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ implies that $f(x, y)$ has a relative maximum at the point (a, b).

Determining Relative Extrema

b. $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ implies that $f(x, y)$ has a relative minimum at the point (a, b).

c. $D(a, b) < 0$ implies that $f(x, y)$ has neither a relative maximum nor a relative minimum at the point (a, b), it has instead a saddle point.

d. $D(a, b) = 0$ implies that the test is inconclusive, so some other technique must be used to solve the problem.

Example 1

Find the relative extrema of the function

$$f(x, y) = x^2 + y^2$$

Solution:

We have $f_x = 2x$ and $f_y = 2y$.

To find the critical points, we set $f_x = 0$ and $f_y = 0$ and solve the resulting system of simultaneous equations

$$2x = 0 \qquad \text{and} \qquad 2y = 0$$

obtaining $x = 0$, $y = 0$, or (0, 0), as the sole critical point.

Next, apply the second derivative test to determine the nature of the critical point (0, 0).

Example 1 – *Solution*

cont'd

We compute $f_{xx} = 2$, $f_{yy} = 2$, and $f_{xy} = 0$,

Thus, $D(x, y) = f_{xx} f_{yy} - f^{2}_{xy} = (2)(2) - (0)^2 = 4$.

We have $D(x, y) = 4$, and in particular, $D(0, 0) = 4$.

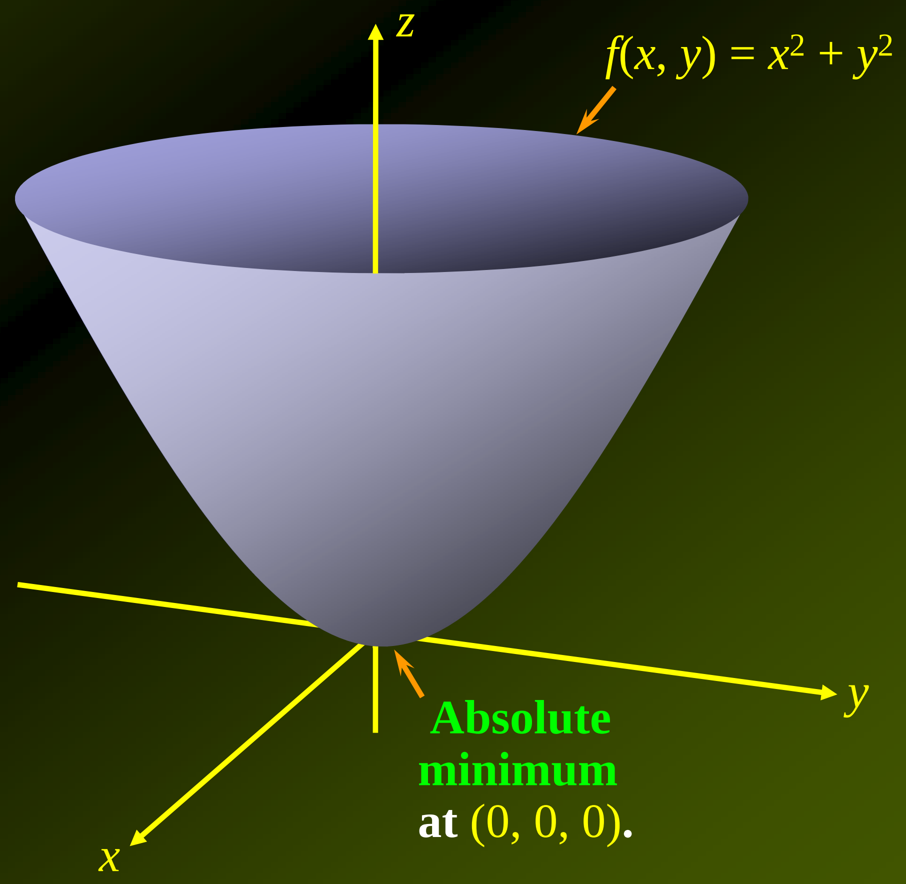
Since $D(0, 0) > 0$ and $f_{xx} = 2 > 0$, we conclude that f has a relative minimum at the point $(0, 0)$.

The relative minimum value, $f(0, 0) = 0$, also happens to be the absolute minimum of f.

Example 1 – *Solution*

cont'd

The relative minimum value, $f(0, 0) = 0$, also happens to be the absolute minimum of f:



Example 2

Find the relative extrema of the function

$$f(x, y) = 3x^2 - 4xy + 4y^2 - 4x + 8y + 4$$

Solution:

We have $f_x = 6x - 4y - 4$ and $f_y = -4x + 8y + 8$

To find the critical points, we set $f_x = 0$ and $f_y = 0$ and solve the resulting system of simultaneous equations

$$6x - 4y - 4 = 0 \quad \text{and} \quad -4x + 8y + 8 = 0$$

obtaining $x = 0$, $y = -1$, or $(0, -1)$, as the sole critical point.

Example 2 – *Solution*

cont'd

Next, apply the second derivative test to determine the nature of the critical point (0, –1).

We compute $f_{xx} = 6$, $f_{yy} = 8$, and $f_{xy} = -4$,

Thus, $D(x, y) = f_{xx} \cdot f_{yy} - f^{2}_{xy} = (6)(8) - (-4)^2 = 32$.

We have $D(x, y) = 32$, and in particular, $D(0, -1) = 32$.

Since $D(0, -1) > 0$ and $f_{xx} = 6 > 0$, we conclude that f has a relative minimum at the point (0, –1).

The relative minimum value, $f(0, -1) = 0$, also happens to be the absolute minimum of f.

Example 3

Find the relative extrema of the function

$$f(x, y) = 4y^3 + x^2 - 12y^2 - 36y + 2$$

Solution:

We have $f_x = 2x$ and $f_y = 12y^2 - 24y - 36$

To find the critical points, we set $f_x = 0$ and $f_y = 0$ and solve the resulting system of simultaneous equations

$$2x = 0 \quad \text{and} \quad 12y^2 - 24y - 36 = 0$$

The first equation implies that $x = 0$, while the second equation implies that $y = -1$ or $y = 3$.

Example 3 – *Solution*

cont'd

Thus, there are two critical points of f: (0, –1) and (0, 3).

To apply the second derivative test, we calculate

$$f_{xx} = 2 \qquad f_{yy} = 24(y - 1) \qquad f_{xy} = 0$$

$$D(x, y) = f_{xx} \cdot f_{yy} - f^2_{xy} = (2) \cdot 24(y - 1) - (0)^2 = 48(y - 1)$$

Apply the second derivative test to the critical point (0, –1):

We have $D(x, y) = 48(y - 1)$.

In particular, $D(0, -1) = 48[(-1) - 1] = -96$.

Example 3 – *Solution*

cont'd

Since $D(0, -1) = -96 < 0$ we conclude that f has a saddle point at $(0, -1)$.

The saddle point value is $f(0, -1) = 22$, so there is a saddle point at $(0, -1, 22)$.

Apply the second derivative test to the critical point $(0, 3)$:

We have $D(x, y) = 48(y - 1)$.

In particular, $D(0, 3) = 48[(3) - 1] = 96$.

Example 3 – *Solution*

cont'd

Since $D(0, -1) = 96 > 0$ and $f_{xx}(0, 3) = 2 > 0$, we conclude that f has a relative minimum at the point $(0, 3)$.

The relative minimum value, $f(0, 3) = -106$, so there is a relative minimum at $(0, 3, -106)$.

Applied Example 3 – *Maximizing Profit*

The total weekly revenue that Acrosonic realizes in producing and selling its loudspeaker system is given by

$$R(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y$$

where *x* denotes the number of fully assembled units and *y* denotes the number of kits produced and sold each week.

The total weekly cost attributable to the production of these loudspeakers is

$$C(x, y) = 180x + 140y + 5000$$

Determine how many assembled units and how many kits should be produced per week to maximize profits.

Applied Example 3 – *Solution*

The contribution to Acrosonic's weekly profit stemming from the production and sale of the bookshelf loudspeaker system is given by

$$P(x,y) = R(x,y) - C(x,y)$$

$$= \left(-\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y\right) - (180x + 140y + 5000)$$

$$= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 100y - 5000$$

Applied Example 3 – *Solution*

cont'd

We have $P(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 100y - 5000$

To find the relative maximum of the profit function *P*, we first locate the critical points of *P*.

Setting P_x and P_y equal to zero, we obtain

$$P_x = -\frac{1}{2}x - \frac{1}{4}y + 120 = 0 \quad \text{and} \quad P_y = -\frac{3}{4}y - \frac{1}{4}x + 100 = 0$$

Solving the system of equations we get $x = 208$ and $y = 64$.

Therefore, *P* has only one critical point at (208, 64).

Applied Example 3 – *Solution*

cont'd

To test if the point (208, 64) is a solution to the problem, we use the second derivative test.

We compute

$$P_{xx} = -\frac{1}{2} \qquad P_{yy} = -\frac{3}{4} \qquad P_{xy} = -\frac{1}{4}$$

So,

$$D(x, y) = \left(-\frac{1}{2}\right)\left(-\frac{3}{4}\right) - \left(-\frac{1}{4}\right)^2 = \frac{3}{8} - \frac{1}{16} = \frac{5}{16}$$

In particular, $D(208, 64) = 5/16 > 0$.

Since $D(208, 64) > 0$ and $P_{xx}(208, 64) < 0$, the point (208, 64) yields a relative maximum of P.

Applied Example 3 – *Solution*

cont'd

The relative maximum at (208, 64) is also the absolute maximum of *P*.

We conclude that Acrosonic can maximize its weekly profit by manufacturing 208 assembled units and 64 kits.

The maximum weekly profit realizable with this output is

$$P(x, y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 100y - 5000$$

$$\begin{aligned} P(208, 64) &= -\frac{1}{4}(208)^2 - \frac{3}{8}(64)^2 - \frac{1}{4}(208)(64) \\ &\quad + 120(208) + 100(64) - 5000 \\ &= \$10,680 \end{aligned}$$