12 TRIGONOMETRIC FUNCTIONS

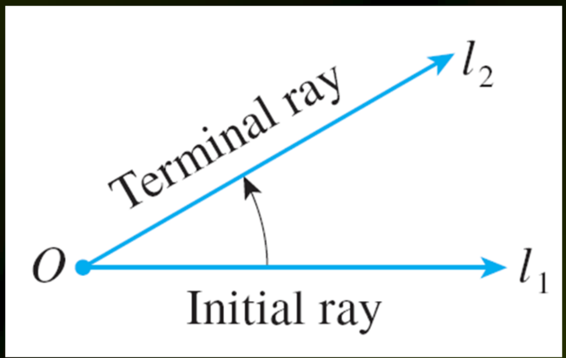
12.1 Measurement of Angles

Angles

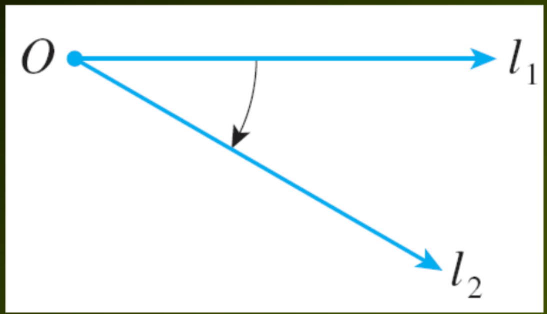
Angles

An angle consists of two rays that intersect at a common endpoint. If we rotate the ray l_1 in a *counterclockwise* direction about the point *O*, the angle generated is *positive* (Figure 1a).

On the other hand, if we rotate the ray l_1 in a *clockwise* direction about the point *O*, the angle generated is *negative* (Figure 1b).



(a) Positive angle



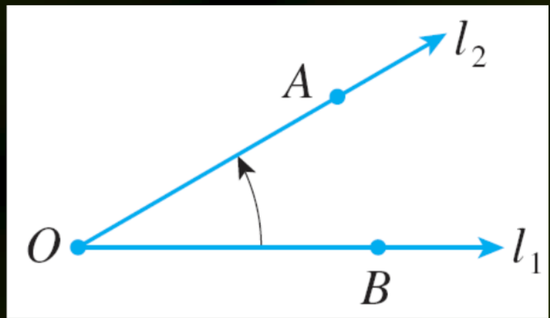
(b) Negative angle

Figure 1

Angles

We refer to the ray l_1 as the initial ray**,** the ray l_2 as the terminal ray**,** and the endpoint *O* as the **vertex** of the angle.

If *A* and *B* are points on l_2 and l_1, respectively, then we refer to the angle as angle *AOB* (Figure 1c).

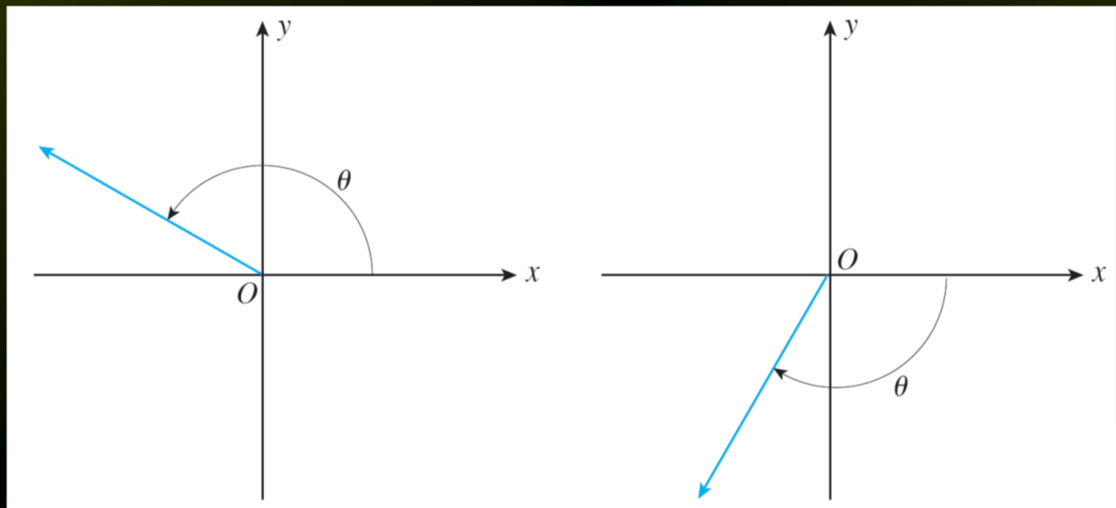


(c) $\angle$ AOB

Figure 1

Angles

We can represent an angle in the rectangular coordinate system. An angle is in **standard position** if the vertex of the angle is centered at the origin and its initial side coincides with the positive *x*-axis (Figures 2a and b).



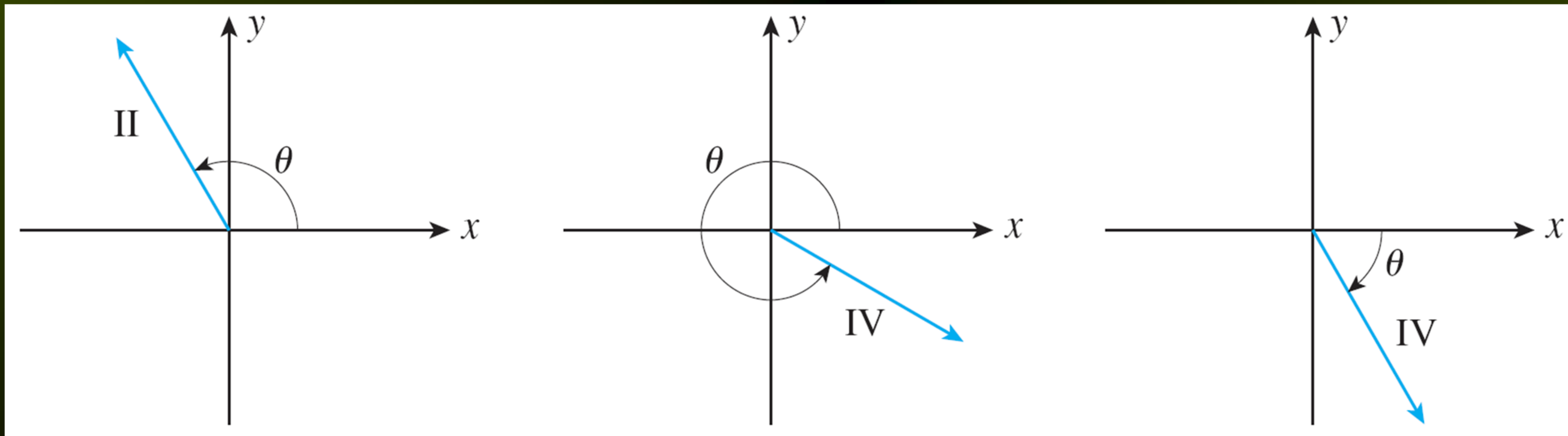
(a) $\theta > 0$

(b) $\theta < 0$

Figure 2

Angles

When we say that an angle lies in a certain quadrant, we are referring to the quadrant in which the terminal ray lies. The angle shown in Figure 3a lies in Quadrant II, and the angles shown in Figures 3b and 3c lie in Quadrant IV.



(a) θ lies in Quadrant II. (b) θ lies in Quadrant IV. (c) θ lies in Quadrant IV.

Figure 3

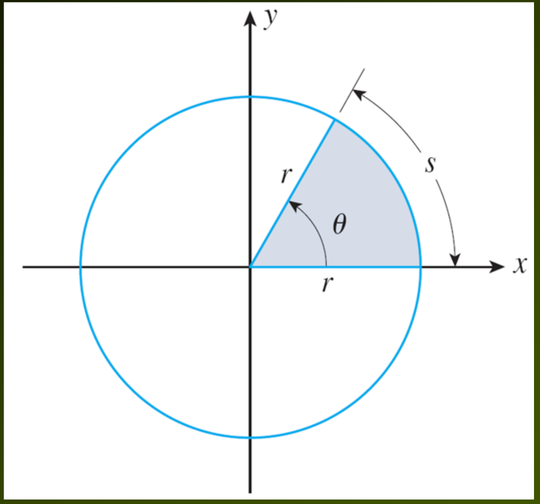
Degree and Radian Measure

Degree and Radian Measure

An angle may be measured in either degrees or radians. A degree is the measure of the angle formed by $\frac{1}{360}$ of one complete revolution. If we rotate an initial ray in standard position through one complete revolution, we obtain an angle of 360°.

A radian is the measure of the central angle subtended by an arc equal in length to the radius of the circle.
In Figure 4, if s is the length of the arc subtended by a central angle θ in a circle of radius r, then

$$\theta = \frac{s}{r} \text{ radians} \qquad (1)$$



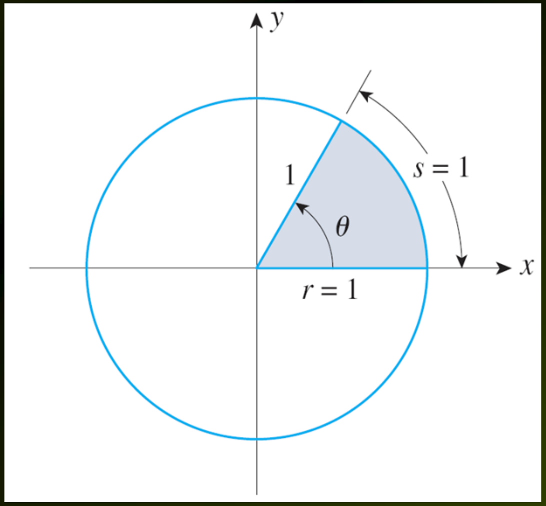
$\theta = \frac{s}{r}$ radians

Figure 4

Degree and Radian Measure

For convenience, we consider a circle of radius 1 centered at the origin. We refer to this circle as the **unit circle**.

Then an angle of 1 radian is subtended by an arc of length 1 (Figure 5).



$$\theta = \frac{s}{r} = 1 \text{ radian}$$

Figure 5

Degree and Radian Measure

The circumference of the unit circle is 2π, and, consequently, the angle subtended by one complete revolution is 2π radians. In terms of degrees, the angle subtended by one complete revolution is 360°. Therefore,

$$360 \text{ degrees} = 2\pi \text{ radians}$$

So

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$$

and

$$x \text{ degrees} = \frac{\pi}{180} x \text{ radians} \qquad \textbf{(2)}$$

Degree and Radian Measure

Since this relationship is linear, we may specify the rule for converting degree measure to radian measure in terms of the following linear function.

Converting Degrees to Radians

$$f(x) = \frac{\pi}{180} x \text{ radians} \tag{3}$$

where x is the number of degrees and $f(x)$ is the number of radians.

Example 1

Convert each angle to radian measure:

a. 30° **b.** 45° **c.** 300° **d.** 450° **e.** -240°

Solution:

Using Formula (3), we have

a. $f(30) = \dfrac{\pi}{180}(30)$ radians, or $\dfrac{\pi}{6}$ radians

b. $f(45) = \dfrac{\pi}{180}(45)$ radians, or $\dfrac{\pi}{4}$ radians

c. $f(300) = \dfrac{\pi}{180}(300)$ radians, or $\dfrac{5\pi}{3}$ radians

Example 1 – *Solution*

cont'd

d. $f(450) = \dfrac{\pi}{180}(450)$ radians, or $\dfrac{5\pi}{2}$ radians

e. $f(?\ 40) = \dfrac{\pi}{180}(?\ 40)$ radians, or $?\ \dfrac{4\pi}{3}$ radians

Degree and Radian Measure

By multiplying both sides of Equation (2) by $\dfrac{180}{\pi}$, we have

$$x \text{ radians} = \frac{180}{\pi}\, x \text{ degrees}$$

Thus, we have the following rule for converting radian measure to degree measure.

Converting Radians to Degrees

$$g(x) = \frac{180}{\pi} x \text{ degrees} \qquad \textbf{(4)}$$

where x is the number of radians and $g(x)$ is the number of degrees.

Degree and Radian Measure

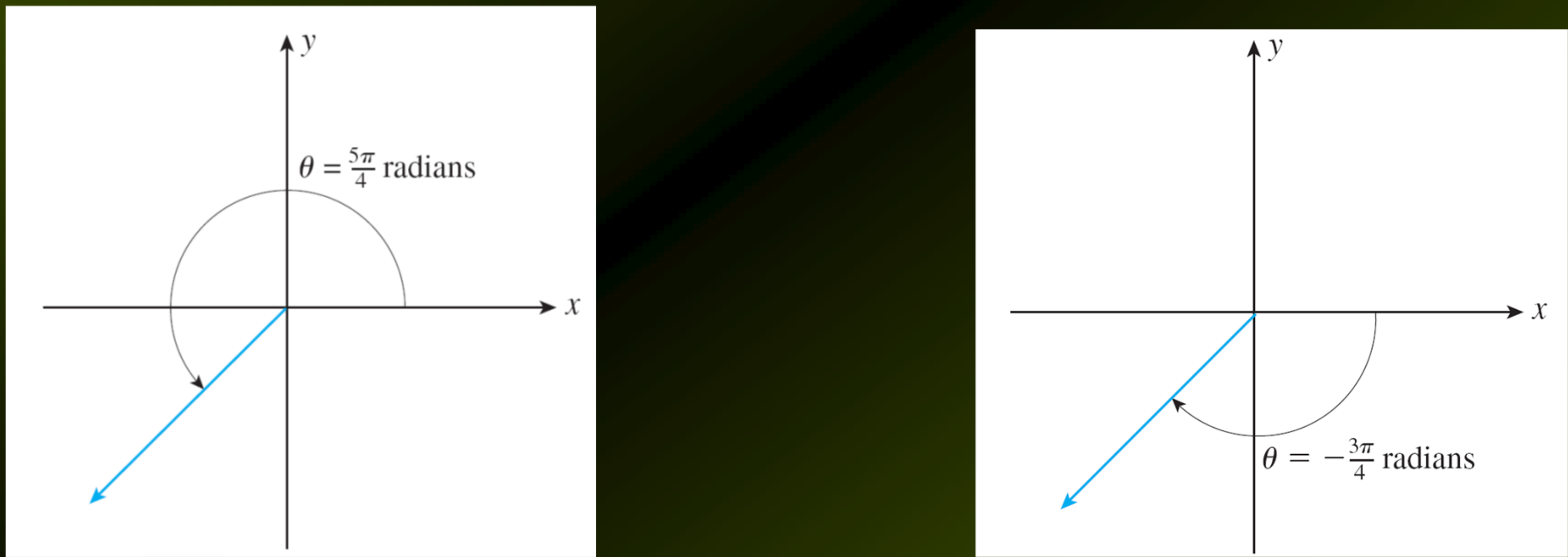
Take time to familiarize yourself with the radian and degree measures of the common angles given in Table 1.

TABLE **1**

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Degree and Radian Measure

You may have observed by now that more than one angle may be described by the same initial and terminal rays. The two angles in Figure 6a and b illustrate this case.



(a) (b)

Coterminal angles

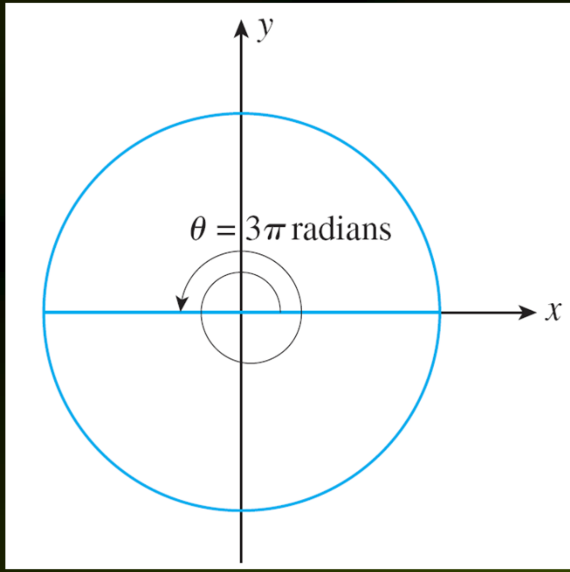
Figure 6

Degree and Radian Measure

The angle $\theta = \frac{5\pi}{4}$ radians is generated by rotating the initial ray in a counter-clockwise direction, and the angle $\theta = -\frac{3\pi}{4}$ radians is generated by rotating the initial ray in a clockwise direction. We refer to such angles as coterminal angles**.**

Degree and Radian Measure

An angle may be greater than 2π radians. For example, an angle of 3π radians is generated by rotating a ray through 1.5 revolutions. As illustrated in Figure 7, an angle of 3π radians has its initial and terminal rays in the same position as an angle of π radians.



$\theta = 3\pi$ radians has its initial and terminal rays in the same position as $\theta = \pi$ radians.

Figure 7

Degree and Radian Measure

A similar statement is true for the negative angles $-\pi$ and -3π radians.

In identifying an angle, we must be careful to specify the *direction* of the rotation and the *number of revolutions* through which the angle has gone.