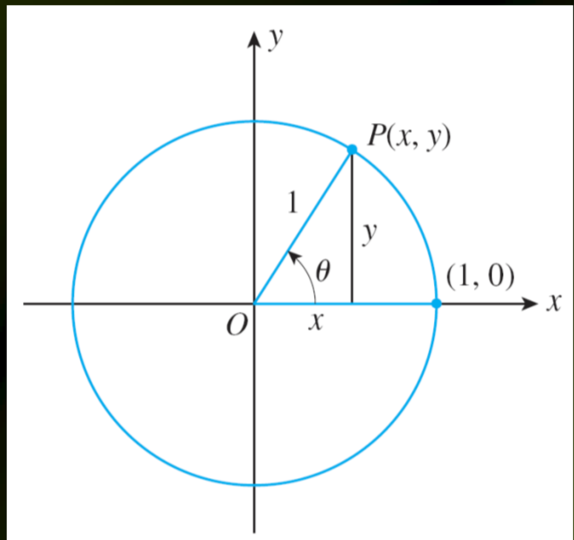
12 TRIGONOMETRIC FUNCTIONS

12.2 The Trigonometric Functions

The Trigonometric Functions and Their Graphs

The Trigonometric Functions and Their Graphs

Let $P(x, y)$ be a point on the unit circle such that the radius $\overline{OP}$ forms an angle of θ radians with respect to the positive x-axis (see Figure 8).



P is a point on the unit circle with coordinates $x = \cos\theta$ and $y = \sin\theta$.

Figure 8

The Trigonometric Functions and Their Graphs

We define the **sine** of the angle θ, written sin θ, to be the *y*-coordinate of *P*.

Similarly, the **cosine** of the angle θ, written cosθ, is defined to be the *x*-coordinate of *P*.

The other trigonometric functions, tangent, cosecant, secant, and cotangent of θ—written tan θ, csc θ, sec θ, and cot θ, respectively—are defined in terms of the sine and cosine functions.

The Trigonometric Functions and Their Graphs

Trigonometric Functions

Let $P(x, y)$ be a point on the unit circle such that the radius $\overline{OP}$ forms an angle of θ radians with respect to the positive x-axis. Then

$$\cos\theta = x \qquad \sin\theta = y$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \qquad (x \neq 0)$$

$$\csc\theta = \frac{1}{y} = \frac{1}{\sin\theta} \qquad (y \neq 0)$$

$$\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta} \qquad (x \neq 0)$$

$$\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta} \qquad (y \neq 0)$$

The Trigonometric Functions and Their Graphs

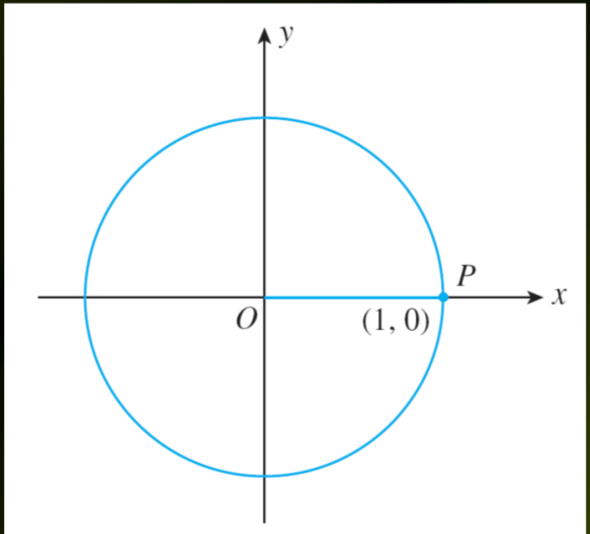
As you work with trigonometric functions, it will be helpful to remember the values of the sine, cosine, and tangent of some important angles, such as $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and so on.

These values may be found using elementary geometry and algebra.

The Trigonometric Functions and Their Graphs

For example, if $\theta = 0$, then the point P has coordinates (1, 0) (see Figure 9a), and we see that

$$\sin 0 = y = 0 \qquad \cos 0 = x = 1 \qquad \tan 0 = \frac{y}{x} = 0$$



(a) $P = (1, 0)$, $\theta = 0$, and $\cos \theta = 1$, $\sin \theta = 0$

Figure 9

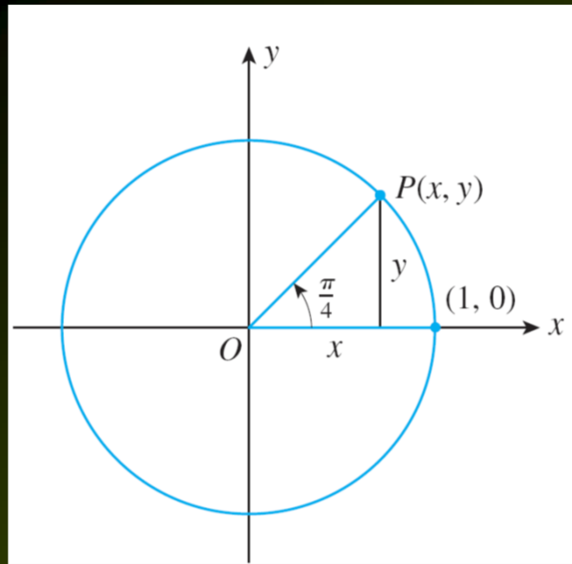
The Trigonometric Functions and Their Graphs

As another example, if $\theta = \dfrac{\pi}{4}$, $x = y$. (See Figure 9b.)

By the Pythagorean theorem, we have

$$x^2 + y^2 = 2x^2 = 1$$

and $x = y = \dfrac{\sqrt{2}}{2}$.



(b) $\theta = \pi/4$, $x = y$, and $\cos\theta = \sin\theta = \sqrt{2}/2$

Figure 9

The Trigonometric Functions and Their Graphs

Therefore,

$$\sin\frac{\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = x = \frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4} = \frac{y}{x} = 1$$

The values of the sine and cosine of some common angles are given in Table 2.

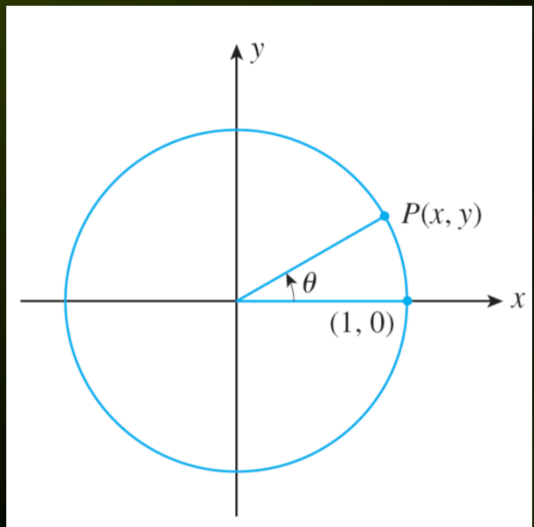
TABLE 2

θ in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{5}{4}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{7}{4}\pi$	$\frac{11}{6}\pi$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

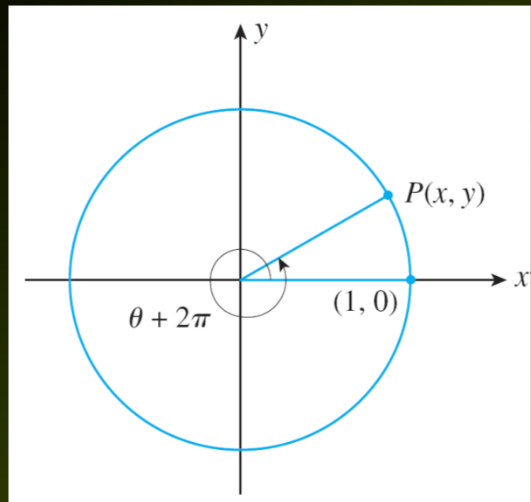
The Trigonometric Functions and Their Graphs

Since the rotation of the radius $\overline{OP}$ by 2π radians leaves it in its original configuration (see Figure 10), we see that

$$\sin(\theta + 2\pi) = \sin\theta \quad \text{and} \quad \cos(\theta + 2\pi) = \cos\theta$$



(a) The angle θ



(b) The angle $(\theta + 2\pi)$

Figure 10

The Trigonometric Functions and Their Graphs

It can be shown that 2π is the smallest positive number for which the preceding equations hold true. That is, the sine and cosine function are periodic with **period 2π.**

It follows that

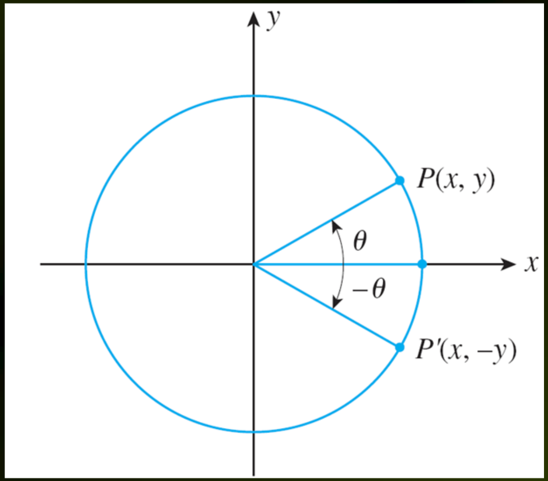
$$\sin(\theta + 2n\pi) = \sin\theta \quad \text{and} \quad \cos(\theta + 2n\pi) = \cos\theta \qquad (5)$$

whenever n is an integer.

The Trigonometric Functions and Their Graphs

Also, from Figure 11, we see that

$$\sin(-\theta) = -\sin\,\theta \qquad \text{and} \qquad \cos(-\theta) = \cos\,\theta \qquad\qquad (6)$$



$\sin(-\theta) = -\sin\,\theta$ and $\cos(-\theta) = \cos\,\theta$.

Figure 11

Example 1

Evaluate

a. $\sin \frac{7\pi}{2}$ b. $\cos 5\pi$ c. $\sin\left(? \frac{5\pi}{2} \right)$ d. $\cos\left(? \frac{11\pi}{4} \right)$

Solution:

a. Using (5) and Table 2, we have

$$\sin\left(\frac{7\pi}{2}\right) = \sin\left(2\pi + \frac{3\pi}{2}\right) = \sin\frac{3\pi}{2} = ?$$

b. Using (5) and Table 2, we have

$$\cos 5\pi = \cos(4\pi + \pi) = \cos \pi = ?$$

Example 1 – *Solution*

cont'd

c. Using (5), (6), and Table 2, we have

$$\sin\left(?\frac{5\pi}{2}\right) = \quad \text{in}\left(\frac{5\pi}{2}\right) = \quad \text{in}\left(2\pi + \frac{\pi}{2}\right) = \quad \text{in}\,\frac{\pi}{2} = ?$$

d. Using (5), (6), and Table 2, we have

$$\cos\left(?\frac{11\pi}{4}\right) = \cos\frac{11\pi}{4} = \cos\left(2\pi + \frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

The Trigonometric Functions and Their Graphs

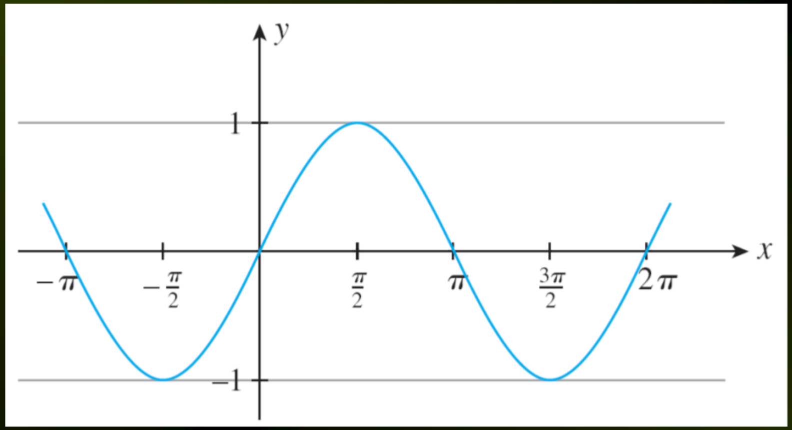
To draw the graph of the function $y = f(x) = \sin x$, we first note that $\sin x$ is defined for every real number x, so the domain of the sine function is $(-\infty, \infty)$.

Next, since the sine function is periodic with period 2π, it suffices to concentrate on sketching that part of the graph of $y = \sin x$ on the interval $[0, 2\pi]$ and repeating it as necessary.

The Trigonometric Functions and Their Graphs

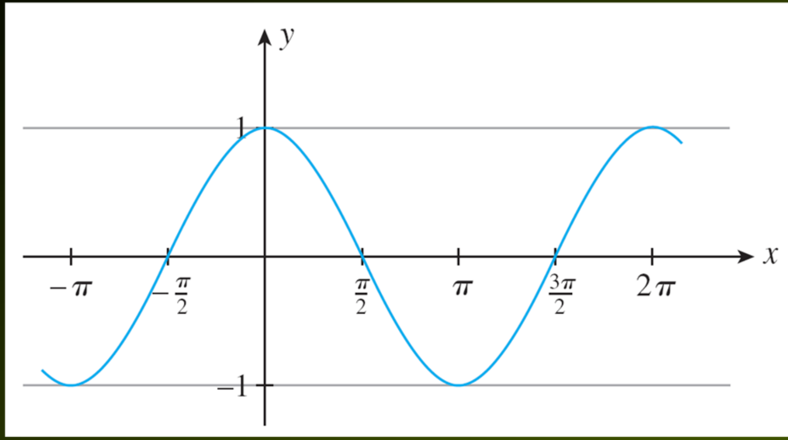
With the help of Table 2, we sketch the graph of $y = \sin x$ (see Figure 12).

In a similar manner, we can sketch the graph of $y = \cos x$ (Figure 13).



The graph of $y = \sin x$

Figure 12



The graph of $y = \cos x$

Figure 13

The Trigonometric Functions and Their Graphs

To sketch the graph of $y = \tan x$, note that $\tan x = \sin x/\cos x$, and so $\tan x$ is not defined when $\cos x = 0$—that is, when

$$x = \frac{\pi}{2} \; ? \; n\pi \quad (n = 0, 1, 2, 3, \ldots).$$

The function is defined at all other points so the domain of the tangent function is the set of all real numbers with the exception of the points just noted.

Next, we can show that the vertical lines with equation

$x = \frac{\pi}{2} \; ? \; n\pi$ ($n = 0, 1, 2, 3, \ldots$) are vertical asymptotes of

the graph of $f(x) = \tan x$.

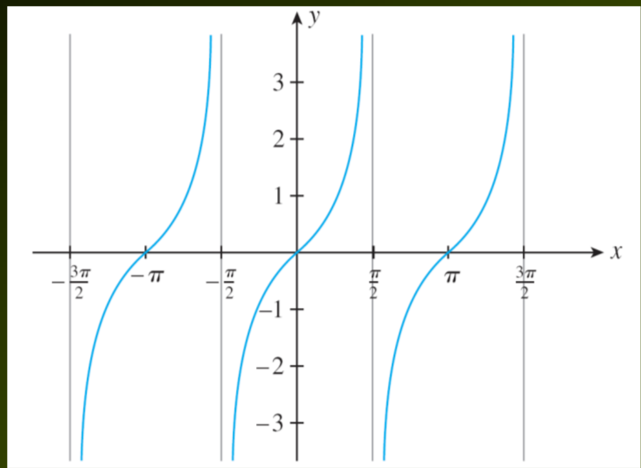
The Trigonometric Functions and Their Graphs

For example, since

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty \quad \text{and} \quad \lim_{x \to \frac{\pi}{2}^+} \tan x = ?\infty$$

which we readily verify with the help of a calculator, we conclude that $x = \frac{\pi}{2}$ is a vertical asymptote of the graph of $y = \tan x$.

Finally, using Table 2, we can sketch the graph of $y = \tan x$ (Figure 14).



The graph of $y = \tan x$

Figure 14

The Trigonometric Functions and Their Graphs

Observe that the tangent function is periodic with period π.

It follows that

$$\tan(x + n\pi) = \tan x \qquad \textbf{(7)}$$

whenever n is an integer.

The graphs of $y = \sec x$, $y = \csc x$, and $y = \cot x$ may be sketched in a similar manner.

The Predator–Prey Population Model

The Predator–Prey Population Model

We will now look at a specific mathematical model of a phenomenon exhibiting cyclical behavior—the so-called predator–prey population model.

Applied Example 2 – *Predator–Prey Population*

The population of owls (predators) in a certain region over a 2-year period is estimated to be

$$p_1(t) = 1000 + 100 \sin\left(\frac{\pi t}{12}\right)$$

in month t, and the population of mice (prey) in the same area at time t is given by

$$p_2(t) = 20{,}000 + 4000 \cos\left(\frac{\pi t}{12}\right)$$

Sketch the graphs of these two functions and explain the relationship between the sizes of the two populations.

Applied Example 2 – *Solution*

We first observe that both of the given functions are periodic with period 24 (months). To see this, recall that both the sine and the cosine functions are periodic with period 2π.

So the period of P_1 and P_2 is obtained by solving the equation

$$\frac{\pi t}{12} = 2\pi$$

giving $t = 24$ as the period of $\sin\left(\frac{\pi t}{12}\right)$.

Applied Example 2 – *Solution*

cont'd

Since $P_1(t + 24) = P_1(t)$, we see that the function P_1 is periodic with period 24.

Similarly, one verifies that the function P_2 is also periodic, with period 24, as asserted.

Next, recall that both the sine and the cosine functions oscillate between –1 and +1, so $P_1(t)$ is seen to oscillate between [1000 + 100(–1)], or 900, and [1000 + 100(1)], or 1100, while $P_2(t)$ oscillates between [20,000 + 4000(–1)], or 16,000, and [20,000 + 4000(1)], or 24,000.

Applied Example 2 – *Solution*

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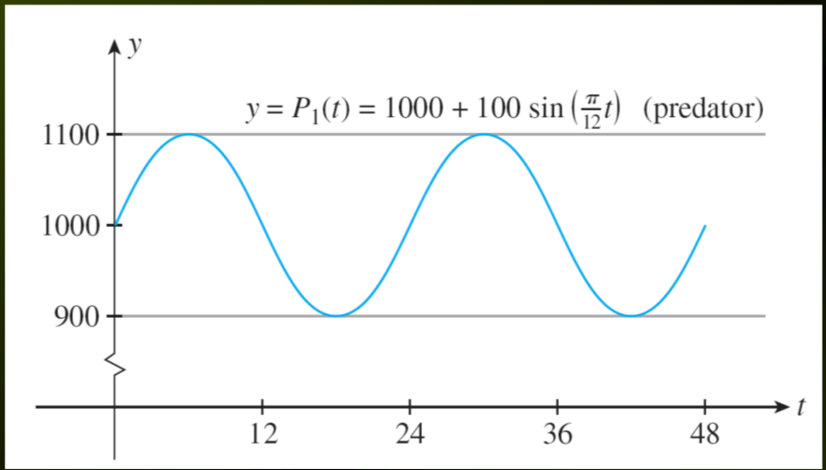
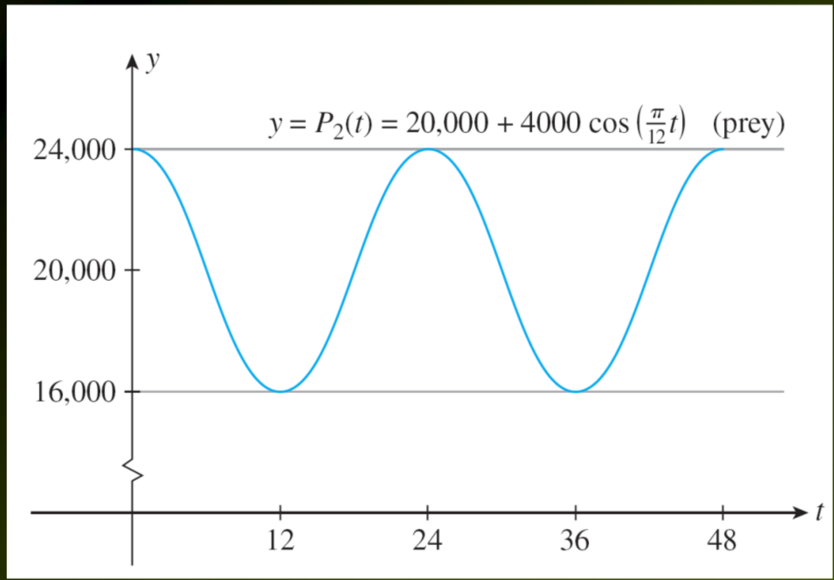
Finally, plotting a few points on each graph for—say, $t = 0, 2, 3$, and so on—we obtain the graphs of the functions P_1 and P_2 as shown in Figure 15.



(a) The graph of the predator function $P_1(t)$



(b) The graph of the prey function $P_2(t)$

Figure 15

Applied Example 2 – *Solution*

cont'd

From the graphs, we see that at time $t = 0$ the predator population stands at 1000 owls.

As it increases, the prey population decreases from 24,000 mice at that instant of time.

Eventually, this decrease in the food supply causes the predator population to decrease, which in turn causes an increase in the prey population.

But as the prey population increases, resulting in an increase in food supply, the predator population once again increases. The cycle is complete and starts all over again.

Trigonometric Identities

Trigonometric Identities

Equations expressing the relationships between trigonometric functions, such as

$$\sin(-\theta) = -\sin\,\theta \qquad \text{and} \qquad \cos(-\theta) = \cos\,\theta$$

are called trigonometric identities.

Trigonometric Identities

Some other important trigonometric identities are listed in Table 3.

TABLE 3

Trigonometric Identities

Pythagorean Identities	**Half-Angle Formulas**	**Sum and Difference Formulas**
$\sin^2\theta + \cos^2\theta = 1$	$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
$\tan^2\theta + 1 = \sec^2\theta$	$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$	$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
$\cot^2\theta + 1 = \csc^2\theta$		

Double-Angle Formulas	**Cofunctions of Complementary Angles**
$\sin 2A = 2 \sin A \cos A$	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
$\cos 2A = \cos^2 A - \sin^2 A$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

Each identity holds true for every value of in the common domain of the specified functions.

Trigonometric Identities

These identities are useful in simplifying trigonometric expressions and equations and in deriving other trigonometric relationships.

They are also used to verify other trigonometric identities, as illustrated in the next example.

Example 3

Verify the identity

$$\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$$

Solution:

We verify this identity by showing that the expression on the left side of the equation can be transformed into the expression on the right side. Thus,

$$\sin\theta(\csc\theta - \sin\theta) = \sin\theta\csc\theta - \sin^2\theta$$

$$= \sin\theta\frac{1}{\sin\theta} ? \sin^2\theta$$

Example 3 – *Solution*

cont'd

$$= 1 - \sin^2\theta$$

$$= \cos^2\theta$$